

Financial derivatives (based on two supports) evaluation

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Abstract

In this paper we build a PDE like Black-Scholes equation in hypothesis of a financial derivative that is dependent on two supports (usual is dependent only on one support), like an option based on gold, when national currency has a great float.

Keywords: Financial derivatives, derivatives evaluation, derivatives based on two supports, extended Itô like lemma.

1. Assuming the model

We suppose that two supports S , T have a generalized Brownian motion:

$$dS = A dt + B dW_t^1 \quad (1)$$

$$dT = C dt + D dW_t^2 \quad (2)$$

where W_t^1 and W_t^2 are two ρ -correlated Wiener process (named sometime as Brownian motion, Wiener-Levy process, Wiener-Bachelier process or Wiener-Einstein process, see [1], p. 123):

$$dW_t^1 dW_t^2 = \rho dt \quad (3)$$

and where A , B , C , D are some two-variables functions:

$$A = A(S, T, t) \quad (4)$$

$$B = B(S, T, t) \quad (5)$$

$$C = C(S, T, t) \quad (6)$$

$$D = D(S, T, t) \quad (7)$$

Remark: In main cases, A and C , respectively B and D have same formulae.

If we have a financial derivative based on these two supports. We denote with $F(S, T, t)$ the evaluation of our derivatives at timestamp t , when current supports level are S and T . If introduce payoff function with fructification of derivative at maturity time (t_{maturity}), than we have a condition on boundary:

$$F(S, T, t_{\text{maturity}}) = \text{payoff}(S, T) \quad (8)$$

Other boundary conditions can be build for zero-value and infinity-value of supports:

$$F(s, 0, t) = F(0, s, t) = 0 \quad (9)$$

$$\lim_{s \rightarrow \infty} F_s(S, T, t) = \lim_{T \rightarrow \infty} F_T(S, T, t) = 1 \quad (10)$$

2. Risk-free portfolio building and Black-Scholes type PDE

Theorem A (*extended Itô-like lemma, see [2], for Itô's lemma see [3]*): Let be S_t and T_t two stochastic precesses defined with next stochastic differential equations:

$$dS = A(S, T, t) dt + B(S, T, t) dW_t^1 \quad (11)$$

$$dT = C(S, T, t) dt + D(S, T, t) dW_t^2 \quad (12)$$

where W_t^1 and W_t^2 are two ρ -correlated Wiener process:

$$dW_t^1 dW_t^2 = \rho dt \quad (13)$$

If $f(S, T, t)$ is a differentiable function, then:

$$df = [f_t + f_s A + f_T C + \frac{1}{2} f_{ss} B^2 + \frac{1}{2} f_{TT} D^2 + f_{ST} \rho B D] dt + [f_s B] dW_1 + [f_T D] dW_2 \quad (14)$$

Proof (see [2]).

Lemma B (risk-free portfolio): If we have a risk-free portfolio $P(t)$, than

$$dP(t) = r P(t) dt \quad (15)$$

where r is the annualized risk-free interest rate.

Proof. Obviously.

Theorem C (*building a risk-free mixt-portfolio with derivatives and supports*): Assuming model (see supra) the portfolio of:

1. one unit of derivative with value $F(S, T, t)$;
2. $-F_s(S, T, t)$ units of S stock with total value $-S F_s(S, T, t)$;
3. $-F_T(S, T, t)$ units of T stock with total value $-T F_T(S, T, t)$,

Than this portfolio is a risk-free portfolio.

Proof. Let a risk-free portfolio P of:

1. one unit of derivative with value $F(S, T, t)$;
2. a units of S stock with total value $a S$;
3. b units of T stock with total value $b T$.

Value of portfolio is:

$$P = F + a S + b T \quad (16)$$

From (16), (14), (1) and (2) we have:

$$\begin{aligned} dP = dF + a dS + b dT = & ([f_t + f_s A + f_T C + \frac{1}{2} f_{ss} B^2 + \frac{1}{2} f_{TT} D^2 + f_{ST} \rho B D] dt + [f_s B] dW_1 + [f_T D] \\ & dW_2) + a (A dt + B dW_1) + b (C dt + D dW_2) = [f_t + f_s A + f_T C + \frac{1}{2} f_{ss} B^2 + \frac{1}{2} f_{TT} D^2 + f_{ST} \rho B D + aA \\ & + bC] dt + [f_s B + aB] dW_1 + [f_T D + bD] dW_2 \end{aligned} \quad (17)$$

From (15), (17) and (16) we have:

$$dP = r P dt = r (F + a S + b T) dt \quad (18)$$

From (17) and (18) we have:

$$f_t + f_S A + f_T C + \frac{1}{2} f_{SS} B^2 + \frac{1}{2} f_{TT} D^2 + f_{ST} \rho B D + a A + b C = r (F + a S + b T) \quad (19)$$

$$f_S B + a B = 0 \quad (20)$$

$$f_T D + b D = 0 \quad (21)$$

From (20) and (21) obtain that

$$a = -f_S \quad (22)$$

$$b = -f_T \quad (23)$$

q.e.d.

Corrolary D (*Black-Scholes-like PDE*) Assuming model (see supra), the value of derivative verify next PDE:

$$f_t + \frac{1}{2} f_{SS} B^2 + \frac{1}{2} f_{TT} D^2 + f_{ST} \rho B D - r F + f_S S + f_T T = 0 \quad (24)$$

Proof. From (19), (22) and (23) we have:

$$0 = f_t + f_S A + f_T C + \frac{1}{2} f_{SS} B^2 + \frac{1}{2} f_{TT} D^2 + f_{ST} \rho B D + a A + b C - r (F + a S + b T) = f_t + f_S A + f_T C + \frac{1}{2} f_{SS} B^2 + \frac{1}{2} f_{TT} D^2 + f_{ST} \rho B D - f_S A - f_T C - r (F - f_S S - f_T T) = f_t + \frac{1}{2} f_{SS} B^2 + \frac{1}{2} f_{TT} D^2 + f_{ST} \rho B D - r F + f_S S + f_T T \quad (25)$$

q.e.d

Remarks E: If $S = T$, $A = C$, $B = D$ and $\rho = 0$ we obtain from (25) generalized Black-Scholes equation:

$$F_t + \frac{1}{2} B^2 F_{SS} + r S F_S - r F = 0. \quad (26)$$

If $A = \mu S$ and $B = \sigma S$ we obtain from (26) Black-Scholes equation (see [4]):

$$F_t + \frac{1}{2} \sigma^2 S^2 F_{SS} + r S F_S - r F = 0. \quad (27)$$

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