

Algorithm for Financial Derivatives Evaluation in Generalized Double-Heston Model

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Abstract

This paper shows how can be estimated the value of an option if we assume the double-Heston model on a message-based architecture. For path trace simulation we will discretize continous model with an Euler division of time.

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1. BSM model, Heston model, Double-Heston model

From physical models, the following situation has reached acceptance: a financial asset interest rate follows a normal law, where the mean is the drift rate and the deviation is the volatility. This leads to a model that is currently accepted in finance; the model of *geometric Brownian motion*. This model (known as *Black–Scholes–Merton model* in finance and financial engineering, see [1]) is a stochastic differential equation (1):

$$dS(t) = m S(t) dt + s S(t) dB(t), \quad (1)$$

where:

- a) $(S(t), t \geq 0)$ is a stochastic process for the value of stock;
- b) m is a static parameter for the drift rate of return;
- c) s^2 is a static parameter for the volatility of stock ($s \geq 0$);
- d) $(B(t), t \geq 0)$ is a standard Wiener process.

Another model is assumed by Heston (see [2]) and it consists from two stochastic differential equations. *The Heston* model corrects some inconsistency of the Black–Scholes–Merton model, for example:

- a) in reality, volatility is not a static parameter; it can be used as static value only on short periods (this value will obtain on calibration process, usual with a statistical estimator);
- b) on long periods, it is possible that interest rate series did not verify a normal law.

The Heston model is described by the following coupled stochastic differential equations (2), (3):

$$dS(t) = A(S(t), v(t), t) dt + B(S(t), v(t), t) dB_1(t) \quad (2)$$

$$dv(t) = C(S(t), v(t), t) dt + D(S(t), v(t), t) dB_2(t) \quad (3)$$

where:

- a) $(S(t), t \geq 0)$ is a stochastic process for value of stock;
- b) $(v(t), t \geq 0)$ is a stochastic process for volatility of value of stock;

- c) $A(S, v, t)$, $B(S, v, t)$, $C(S, v, t)$, $D(S, v, t)$ are three parametric algebraic functions;
d) $(B_1(t), t \geq 0)$ and $(B_2(t), t \geq 0)$ are two r -correlated standard Wiener processes, i.e. (4):

$$dB_1(t) dB_2(t) = r dt \quad (4)$$

For Wiener processes, more details can be found in [3]. For the basic Heston model we have (5):

$$\left. \begin{array}{l} \text{a) } A = S(t) m \\ \text{b) } B = S(t) v(t) \\ \text{c) } C = K (\theta - v(t)) \\ \text{d) } D = \xi v(t) \end{array} \right\} \quad (5)$$

where:

- a) m is a drift of rate;
b) θ is long run average price volatility; as t tends to infinity, the expected value of $v(t)$ tends to θ ;
c) K is the rate at which $v(t)$ reverts to θ ;
d) ξ is the volatility of the volatility; as the name suggests, this determines the variance of $v(t)$.

Note that for $C = D = 0$ we obtain a static volatility model (Black–Scholes–Merton) (6)

$$dv(t) = 0.$$

The Double-Heston model (see [4]) is described by the following coupled stochastic differential equations (7), (8), (9):

$$dS(t) = M(S(t), v_1(t), v_2(t), t) dt + S_1(S(t), v_1(t), t) dB_1(t) + S_2(S(t), v_2(t), t) dB_2(t) \quad (7)$$

$$dv_1(t) = C_1(S(t), v_1(t), t) dt + D_1(S(t), v_1(t), t) dB_3(t) \quad (8)$$

$$dv_2(t) = C_2(S(t), v_2(t), t) dt + D_2(S(t), v_2(t), t) dB_4(t) \quad (9)$$

where:

- a) $(S(t), t \geq 0)$ is a stochastic process for value of stock;
b) $(v_1(t), t \geq 0)$ is a stochastic process for *half*-volatility of value of stock;
c) $(v_2(t), t \geq 0)$ is a stochastic process for *half*-volatility of value of stock;
d) $M(S, v_1, v_2, t)$, $S_1(S, v_1, t)$, $S_2(S, v_2, t)$, $C_1(S, v_1, t)$, $D_1(S, v_1, t)$, $C_2(S, v_2, t)$, $D_2(S, v_2, t)$ are three/four parametric algebraic functions;
e) $(B_1(t), t \geq 0)$ and $(B_3(t), t \geq 0)$ are two r_1 -correlated standard Wiener processes, i.e. (10):

$$dB_1(t) dB_3(t) = r_1 dt \quad (10)$$

- f) $(B_2(t), t \geq 0)$ and $(B_4(t), t \geq 0)$ are two r_2 -correlated standard Wiener processes, i.e. (11):

$$dB_2(t) dB_4(t) = r_2 dt \quad (11)$$

- g) $(B_1(t), t \geq 0)$ and $(B_2(t), t \geq 0)$ are two independent standard Wiener processes.

For the basic Double-Heston model we have (12):

$$\left. \begin{array}{l} \text{a) } M = S(t) m \\ \text{b) } S_1 = S(t) v_1(t) \\ \text{c) } S_2 = S(t) v_2(t) \\ \text{d) } C_1 = K_1 (\theta_1 - v_1(t)) \\ \text{e) } C_2 = K_2 (\theta_2 - v_2(t)) \\ \text{f) } D_1 = \xi_1 v_1(t) \\ \text{g) } D_2 = \xi_2 v_2(t) \end{array} \right\} \quad (12)$$

where:

- a) m is a drift of rate;
- b) θ_1 is long run average price volatility; as t tends to infinity, the expected value of $v_1(t)$ tends to θ_1 ;
- c) θ_2 is long run average price volatility; as t tends to infinity, the expected value of $v_2(t)$ tends to θ_2 ;
- d) K_1 is the rate at which $v_1(t)$ reverts to θ_1 ;
- e) K_2 is the rate at which $v_2(t)$ reverts to θ_2 ;
- f) ξ_1 is the volatility of the volatility; as the name suggests, this determines the variance of $v_1(t)$;
- g) ξ_2 is the volatility of the volatility; as the name suggests, this determines the variance of $v_2(t)$.

Any financial derivative based on support with price $S(t)$ at time t , with quotation at time t and a value S of support as $V(S, v_1, v_2, t)$, where (12):

$$V : R_+ \times [0, T] \times [0, T] \times [0, T] \rightarrow R_+ \quad (12)$$

and at maturity time T will generate an generate an *payoff* (13):

$$\text{payoff}: R_+ \rightarrow R_+$$

For example, *European options CALL and PUT* has payoff functions (14):

$$\left. \begin{array}{l} \text{payoff}(x) = \max\{0, x - E\} \\ \text{payoff}(x) = \max\{0, E - x\} \end{array} \right\} \quad (14)$$

where E is *exercise price of option*.

2. Path trace simulation for option's pricing in generalized Double-Heston model

First, we discretize continuous dimension of time. Let us denote (15):

$$t[k] = t[0] + k\Delta, \quad 0 \leq k \leq N \quad (15)$$

where:

- a) $\Delta = (T - t[0]) / N$
- b) T is the maturity time of option;
- c) N is a number of time units (like days, hours, minutes, etc); note that sometimes is used transaction days - in this case, discretization hasn't a constant step.

Because for a standard Wiener process $(B(t), t \geq 0)$ we can obtain a standard normal random variable series $(X[B(t)], t \geq 0)$ with (16):

$$dB(t) = X(dt)^{1/2} \quad (16)$$

we can build a simulation step as (17):

$$\left. \begin{aligned} M &\leftarrow M(S[k], v_1[k], v_2[k], t[k]) \\ S_1 &\leftarrow S_1(S[k], v_1[k], t[k]) \\ S_2 &\leftarrow S_2(S[k], v_2[k], t[k]) \\ C_1 &\leftarrow C_1(S[k], v_1[k], t[k]) \\ D_1 &\leftarrow D_1(S[k], v_1[k], t[k]) \\ C_2 &\leftarrow C_2(S[k], v_2[k], t[k]) \\ D_2 &\leftarrow D_2(S[k], v_2[k], t[k]) \\ S[k+1] &\leftarrow S[k] + M\Delta + S_1 X_1 \sqrt{\Delta} + S_2 X_2 \sqrt{\Delta} \\ v_1[k+1] &\leftarrow v_1[k] + C_1 \Delta + D_1 X_3 \sqrt{\Delta} \\ v_2[k+1] &\leftarrow v_2[k] + C_2 \Delta + D_2 X_4 \sqrt{\Delta} \end{aligned} \right\} \quad (17)$$

where X_1 and X_3 are r_1 -correlated, X_2 and X_4 are r_2 -correlated. A simple method to generate two r correlated normal values is (18):

$$\left. \begin{aligned} X &\leftarrow \text{NormRand}() \\ Z &\leftarrow \text{NormRand}() \\ Y &\leftarrow r X + \sqrt{1-r^2} Z \end{aligned} \right\} \quad (18)$$

where *NormRand* is a function that produces independent real random numbers between 0 and 1, with normal distribution.

A complete simulation for interval $[t_0, T]$ in N steps with evaluation of payoff is function *Simulation*, described below:

```

FUNCTION Simulation()
    S ← S0
    v1 ← v10
    v2 ← v20
    t ← t0
    Δ ← (T – t0) / N
    FOR k ← 1, N
        t ← t + Δ
        X1 ← NormRand()
        X2 ← NormRand()
        Y3 ← NormRand()
        Y4 ← NormRand()
        X3 ← r1 X1 + √(1 – r12) Y3
        X4 ← r2 X2 + √(1 – r22) Y4
        M ← M(S, v1, v2, t)
        S1 ← S1(S, v1, t)

```

```

    S2 ← S2(S, v2, t)
    C1 ← C1(S, v1, t)
    D1 ← D1(S, v1, t)
    C2 ← C2(S, v2, t)
    D2 ← D2(S, v2, t)
    SS ← S + M Δ + S1 X1 √Δ + S2 X2 √Δ
    vv1 ← v1 + C1 Δ + D1 X3 √Δ
    vv2 ← v2 + C2 Δ + D2 X4 √Δ
    S ← SS
    v1 ← vv1
    v2 ← vv2
  END FOR
  RETURN payoff(S)
END FUNCTION

```

3. Monte Carlo method for Option's Pricing in Double-Heston Model

Because for a level of acceptance α , where $0 < \alpha < 1$, a trust interval for $E[S(T)]$ is $[s - a, s + a]$, with (19):

$$s = [Simulation() + Simulation() + \dots + Simulation()] / N \quad (19)$$

and (20):

$$a = F(\alpha/2)\sigma/M^{1/2} \quad (20)$$

where:

- a) N is number of simulations;
- b) F is the inverse function for CDF (cumulative distribution function) of standard normal distribution; it means that (21) or (22):

$$Prob(s - a < E[payoff(S(T))] < s + a) = 1 - \alpha \quad (21)$$

$$Prob(E[payoff(S(T))] = s + O(M^{1/2})) = 1 - \alpha. \quad (22)$$

where big-O notation is a Buchmann-Landau symbol (see [5]). Algorithm for evaluation of $E[payoff(S(T))]$ is described below, in *Serial_Simulation* function:

```

FUNCTION Serial_Simulation()
  LOCAL x
  x ← 0
  FOR i ← 1, M
    x ← x + Simulation()
  ENDFOR
  RETURN x/M
END FUNCTION

```

4. Further works

Like in [6] we will to parallelize Monte Carlo algorithm for generalized Double-Heston model. Also, we want to build a Merton-Garman like PDE for option pricing like in [1] for generalized Double-Heston model, and build some parallelization of PDE numerical solving, like in [4].

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