Black-Scholes formula - a Heston approach

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Abstract
In this paper we will compare Black-Scholes formula with a particular case of Heston formula, both solutions of the same problem.

Keywords: Black-Scholes model, Heston model, comparing analytic solutions.

1. Introduction
As an extension of Black and Scholes (1973) model:

\[ dS = \mu S dt + \sigma S^{0.5} dW, \quad (1) \]

where

a) \( W \) is an Wiener process;
b) \( \mu \) is a constant named drift;
c) \( \sigma \) is a constant named volatility;
d) \( S \) is a process for a traded asset.

Steven and Heston (1993) define a new model with a stochastic volatility, see equation (2):

\[ dS = \mu S dt + v^{0.5} S dW \\
\[ dv = \theta (\sigma - v) dt + \xi v^{0.5} dB, \quad (2) \]

where:

a) \( \omega \) is long term of volatility;
b) \( \theta \) is return factor to mean of volatility (\( \sigma \));
c) \( \xi \) is volatility of volatility;
d) \( B \) and \( W \) are Wiener standard processes \( \rho \)-correlated;
e) \( S \) is a stochastic process for a traded asset;
f) \( v \) is a stochastic process for volatility.

This model was extended by Christoffersen, Heston and Jacobs (2009) as a model with two stochastic semi-volatilities. In our opinion, this model can be generalized as a stochastic model with \( q (q>0) \) stochastic partial-(or semi-)volatilities.

2. Solutions of tho models in mirror

<table>
<thead>
<tr>
<th>SDE for BS model</th>
<th>Analytic solutions for european calls</th>
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</table>
| \[ dS = \mu S dt + \sigma^{0.5} S dW \] | \[ V(s,t) = N(d_1) S - N(d_2) E \exp(-r (T - t)) \]
| \[ d_1 = \sigma^{-1} (T - t)^{0.5} [\ln(S/E) + (r + \frac{1}{2} \sigma^2)(T - t)] \] | \[ d_1 = \sigma^{-1} (T - t)^{0.5} [\ln(S/E) + (r + \frac{1}{2} \sigma^2)(T - t)] \]
| \[ d_2 = \sigma^{-1} (T - t)^{0.5} [\ln(S/E) + (r - \frac{1}{2} \sigma^2)(T - t)] \] | \[ d_2 = \sigma^{-1} (T - t)^{0.5} [\ln(S/E) + (r - \frac{1}{2} \sigma^2)(T - t)] \]
with E strike with BS model

$N(x) - \text{pdf of } N(0,1) \text{ distribution}$

Official references for model and analytic solutions for BS model

(Black, F. & Scholes, M., 1973)

SDEs for H model

$dS = \mu S dt + v^{0.5} S dW$
$dv = \theta (\sigma - v) dt + \xi v^{0.5} dB$

Analytic solutions for European calls with E strike with H model

$V(s,v,t) = S P_1 - E \exp(-r(T-t)) P_2$
$P_j = \frac{1}{2} \left[ \frac{\exp(i \varphi \log(E)) f_j(x,v,t,\varphi)}{\varphi} \right] d\varphi$
$f_j(x,v,t,z) = \exp(C_j(T-t,z) + D_j(T-t,z) z + i z x)$
$C_j(t,z) = r z i t + a \left[ (b_j - \rho \xi z i + d_j) t - 2 \log(1 - g_j \exp(d_j r)) + 2 \log(1 - g_j) \right] \xi^2$
$D_j(t,z) = [(b_j - \rho \xi z i + d_j) [1 - \exp(d_j r)] \xi^2 [1 - \exp(d_j r)]]^{-1}$
$g_j = (b_j - \rho \xi z i + d_j) [1 - \exp(d_j r)]^{-1}$
$d_j = [(b_j - \rho \xi z i) \xi^2 - \xi^2 (2 u_j z i - \xi^2)]^{0.5}$
$u_1 = \frac{1}{2}$
$u_2 = -\frac{1}{2}$
$a = k \theta$
$b_1 = k + \lambda - \rho \xi$
$b_2 = k + \lambda$
$j = 1, 2$

Official references for model and analytic solutions for H model

(Steven & Heston, 1993)

3. Links between solutions?
We can point that Heston model is a generalization of Black-Scholes model. For

$v = \sigma \quad (3)$

id est:

$dv = 0 \quad (4)$

the two models are identical.

But

$dv = 0 \quad (5)$

is same with

$\theta = \xi = 0, \quad (6)$

that means:
\[ a = k \theta = 0, \quad (7) \]

\[ b_1 = k + \lambda - \rho \xi = k + \lambda = b_2, \quad (8) \]

\[ d_j = ((b_j - \rho \xi z i)^2 - \xi^2 (2 u_j z i - \xi^2))^0.5 = ((b_j - \rho \xi z i)^2 + \rho^2)^0.5 = |b_j|, \quad (9) \]

\[ g_j = [b_j - \rho \xi z i + d_j] [b_j - \rho \xi z i - d_j]^{-1} = [b_j + |b_j|] [b_j - |b_j|]^{-1} = 0, \quad (10) \]

if assume that

\[ k + \lambda < 0 \]

\[ D_j(t, z) = [(b_j - \rho \xi z i + d_j) [1 - \exp(d_j r)] \xi^2 [1 - \exp(d_j r)]^{-1} = [b_j + |b_j|] [1 - \exp(|b_j| r)] \xi^2 [1 - \exp(|b_j| r)]^{-1} = 0, \quad (11) \]

if assume that

\[ [b_j + |b_j|] \xi^2 = 0/0 = 0, \quad (12) \]

\[ C_j(t, z) = r z i t + a [(b_j - \rho \xi z i + d_j) t - 2 \log(1 - g_j \exp(d_j r)) + 2 \log(1 - g_j)] \xi^2 = r z i t, \quad (13) \]

if assume that

\[ + a \xi^2 = 0/0 = 0. \quad (14) \]

\[ f_j(x, v, t, z) = \exp(C_j(T - t, z) + D_j(T - t, z) z + i z x) = \exp(r z i t + i z x) \]

\[ P_j = \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\infty} \text{Re} \left[ \frac{\exp(i \varphi \log(E)) f_j(x, v, t, \varphi)}{i \varphi} \right] d\varphi = \frac{1}{2} + \int_{0}^{\infty} \text{Re} \left[ \exp(i z \log(E)) \exp(r z i t + i z x) \right] dz \]

\[ \int_{0}^{\infty} \text{Re} \left[ \cos(z \log(E)) \sin(r z t + z x) + \sin(z \log(E)) \cos(r z t + z x) \right] dz \]

\[ \int_{0}^{\infty} \text{Re} \left[ \sin(z \log(E) + r z t + z x) \right] z^{-1} dz \quad (15) \]

4. Comments and further works

We expected that the two solutions are identical. Because not getting the same result on the two different routes, results that Heston solution has a little inconsistency on some particular cases, like \( \xi = 0 \) (we use that \( 0/\xi = 0! \)). Therefore, a revision of the solution Heston by treating individual cases. We intend to do so in the future.

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