

About Multi-Heston SDE Discretization

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Abstract: in this paper we show how can estimate a financial derivative based on a support if assume for the support a Multi-Heston model.

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1. Heston model for an asset and it's transformation

A new model assumed by Heston consists from two stochastic differential equations for a traded asset (see [1]):

$$(1) \quad dS_t = S_t \sqrt{V_t} dW1_t$$

where the volatility term is stochastic:

$$(2) \quad dV_t = K (\theta - V_t) dt + \varepsilon \sqrt{V_t} dW2_t$$

and K, θ, ε are positive constants, $W1_t$ and $W2_t$ are ρ -correlated:

$$(3) \quad dW1_t dW2_t = \rho dt$$

An Euler-Maruyama discretization (see [2]) of (1) and (2) will be produce something like:

(4.0) Begin

$$(4.1) \quad \Delta = t[k+1] - t[k]$$

$$(4.2) \quad X[k] = \text{NormalValue}()$$

$$(4.3) \quad Z = \text{NromalValue}()$$

$$(4.4) \quad Y[k] = \rho * X[k] + Z * \sqrt{1 - \rho * \rho}$$

$$(4.5) \quad S[k+1] = S[k] + S[k] * \sqrt{V[k]} * X[k] * \sqrt{\Delta}$$

$$(4.6) \quad V[k+1] = V[k] + K(\theta - V[k])\Delta + \varepsilon * \sqrt{V[k]} * Y[k] * \sqrt{\Delta}$$

(4.7) End

Applying Ito's lemma (see [3]), (1) will be:

$$(5) \quad d(\ln S_t) = -V(t) dt / 2 + \sqrt{V_t} dW1_t$$

that will produce after Euler-Maruyama discretization of (5) and (2) produce something like:

(6.0) Begin

$$(6.1) \quad \Delta = t[k+1] - t[k]$$

$$(6.2) \quad X[k] = \text{NormalValue}()$$

$$(6.3) \quad Z = \text{NormalValue}()$$

- (6.4) $Y[k] = \rho * X[k] + Z * \sqrt{1 - \rho^2}$
- (6.5) $LNS[k+1] = LNS[k] - V[k] * \Delta / 2 + \sqrt{V[k]} * X[k] * \sqrt{\Delta}$
- (6.6) $V[k+1] = V[k] + K(\theta - V[k])\Delta + \varepsilon * \sqrt{V[k]} * Y[k] * \sqrt{\Delta}$
- (6.7) End

Comparing complexity of (4.5) and (6.5):

- (7) $C(4.5) = 4 * C(*) + C(\text{sqrt})$
- (8) $C(6.5) = C(+) + C(-) + 3 * C(*) + C(/2) + C(\text{sqrt})$

If suppose that for float numbers we have:

- (9) $C(+) + C(-) + C(/2) \ll C(*)$

than, if number of nodes in a discretized time interval is less by $N_{\max}$:

- (10) $N_{\max} = C(\text{exp}) / (C(*) - C(+) - C(-) - C(/2))$

the modified version of Heston's discretization will be preferred.

2. Double-Heston model for an asset and its discretization

In [4], Heston model is extent to a two stochastic semivolatilities like:

- (11.0) Begin
- (11.1) $dS_t = S_t \sqrt{V_{1t}} dW_{1t} + S_t \sqrt{V_{2t}} dW_{2t}$
- (11.2) $dV_{1t} = K_1 (\theta_1 - V_{1t}) dt + \varepsilon_1 \sqrt{V_{1t}} dW_{2t}$
- (11.3) $dV_{2t} = K_2 (\theta_2 - V_{2t}) dt + \varepsilon_2 \sqrt{V_{2t}} dW_{2t}$
- (11.4) $dW_{1t} dW_{2t} = \rho_1 dt$
- (11.5) $dW_{1t} dW_{2t} = \rho_2 dt$
- (11.6) End

with same rules for K_i , θ_i , ε_i , and W_{1t} and W_{2t} are ρ_i -correlated. Discretization code with Euler scheme is:

- (12.0) Begin
- (12.1) $\Delta = t[k+1] - t[k]$
- (12.2) $X1[k] = \text{NormalValue}()$
- (12.3) $X2[k] = \text{NormalValue}()$
- (12.4) $Z1 = \text{NormalValue}()$
- (12.5) $Z2 = \text{NormalValue}()$
- (12.6) $Y1[k] = \rho_1 * X1[k] + Z1 * \sqrt{1 - \rho_1^2}$
- (12.7) $Y2[k] = \rho_2 * X2[k] + Z2 * \sqrt{1 - \rho_2^2}$
- (12.8) $S[k+1] = S[k] + S[k] * \sqrt{V1[k]} * X1[k] * \sqrt{\Delta}$
 $+ S[k] * \sqrt{V2[k]} * X2[k] * \sqrt{\Delta}$
- (12.9) $V1[k+1] = V1[k] + K1(\theta1 - V1[k])\Delta + \varepsilon1 * \sqrt{V1[k]} * Y1[k] * \sqrt{\Delta}$
- (12.10) $V2[k+1] = V2[k] + K2(\theta2 - V2[k])\Delta + \varepsilon2 * \sqrt{V2[k]} * Y2[k] * \sqrt{\Delta}$
- (12.11) End

After applying an Ito's like lemma on (11.1) obtaining an alternative:

$$(13) \quad d(\ln S_t) = -(V_1(t) + V_2(t)) dt / 2 + \sqrt{V_1(t)} dW_{1t} + \sqrt{V_2(t)} dW_{2t}$$

that will imply next discretization code:

```
(14.0) Begin
(14.1) Δ = t[k+1]-t[k]
(14.2) X1[k] = NormalValue()
(14.3) X2[k] = NormalValue()
(14.4) Z1 = NormalValue()
(14.5) Z2 = NormalValue()
(14.6) Y1[k] = ρ1*X1[k]+Z1*sqrt(1-ρ1*ρ1)
(14.7) Y2[k] = ρ2*X2[k]+Z2*sqrt(1-ρ2*ρ2)
(14.8) LNS[k+1] = LNS[k] - (V1[k]+V2[k])*Δ/2 + sqrt(V1[k])*X1[k]*sqrt(Δ)
      + sqrt(V2[k])*X2[k]*sqrt(Δ)
(14.9) V1[k+1] = V1[k] + K1(θ1-V1[k])Δ + ε1*sqrt(V1[k])*Y1[k]*sqrt(Δ)
(14.10) V2[k+1] = V2[k] + K2(θ2-V2[k])Δ + ε2*sqrt(V2[k])*Y2[k]*sqrt(Δ)
(14.11) End
```

3. Multi-Heston model for an asset and it's discretization

For a Multi-Heston model:

$$(15.1) \quad dS_t = S_t \sqrt{V_{1t}} dW_{1t} + \dots + S_t \sqrt{V_{pt}} dW_{1pt}$$

$$(15.2) \quad dV_{it} = K_i (\theta_i - V_{it}) dt + \varepsilon_i \sqrt{V_{it}} dW_{2it}, i=1,p$$

$$(15.3) \quad dW_{1it} dW_{2it} = \rho_i dt, i=1,p$$

Before and after modification of (15.1) we have:

```
(16.0) Begin
(16.1) Δ = t[k+1]-t[k]
(16.2) Tt := 0
(16.3) For i:=1 to p do
(16.4)   X = NormalValue()
(16.5)   Z = NormalValue()
(16.6)   Y = ρ[i]*X+Z*sqrt(1-ρ[i]*ρ[i])
(16.7)   Tt += sqrt(V[i][k])*X
(16.8)   V[i][k+1] = V[i][k] + K[i](θ[i]-V[i][k])Δ + ε[i]*sqrt(V[i][k])*Y*sqrt(Δ)
(16.9) EndFor
(16.10) S[k+1] = S[k] * (1 + Tt) * sqrt(Δ)
(16.11) End
```

and:

```
(17.0) Begin
(17.1) Δ = t[k+1]-t[k]
(17.2) Vv := 0
(17.3) Tt := 0
(17.4) For i:=1 to p do
(17.5)   X = NormalValue()
(17.6)   Z = NormalValue()
(17.7)   Y = ρ[i]*X+Z*sqrt(1-ρ[i]*ρ[i])
```

```
(17.8)  Vv += V[i][k]
(17.9)  Tt += sqrt(V[i][k])*X
(17.10) V[i][k+1] = V[i][k] + K[i](θ[i]-V[i][k])Δ + ε[i]*sqrt(V[i][k])*Y*sqrt(Δ)
(17.11) EndFor
(17.12) LNS[k+1] = LNS[k] - Vv*Δ/2 + Tt*sqrt(Δ)
(17.13) End
```

4. Further works

We want to build some modifications of (17) as an modified IJK, BK, and TG scheme (see [5]) built on Millstein discretization scheme.

References

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