On Stability of the Mechanical Lagrangian Systems

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Abstract. *We consider MLS (mechanical Lagrangian systems) with external forces. We give some conditions of stability and dissipativity and show that the energy of the system decreases on the integral curves.*

Key words: LMS, stability, dissipative system.

1. Introduction.

We consider a Lagrangian system

$$\sum = (M, L(x,y), F(x,y))$$

with M a differential manifold, L a regular Lagrangian and $F(x,y)$ external forces seen as d-covector field on TM.

The evolution equations of $\sum$ are:

$$\text{(1.1)} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = F_i(x,y),\ y^i = \frac{dx^i}{dt}.$$

Expanding the derivative with respect to t, replacing the derivates $\frac{\partial^2 L}{\partial y^i \partial y^j}$ with $2g_{ij}$ and multipling with (g^{jk}), the equation (1.1) becomes system of differential equations of second order:

$$\text{(1.2.)} \qquad \frac{d^2 x^i}{dt^2} + 2G^i(x,\dot{x}) = \frac{1}{2}F^i(x,\dot{x})$$

where

$$G^i(x,y) = \frac{1}{4} g^{ij}\left(\frac{\partial^2 L}{\partial y^j \partial x^k} y^k - \frac{\partial L}{\partial x^i}\right), y = \dot{x} = \frac{dx^i}{dt}$$

with the notation $y^i = \frac{dx^i}{dt}$, equations (1.2.) are equivalent with the system of equations:

$$\frac{dx^i}{dt} = y^i$$

(1.3.)

$$\frac{dy^i}{dt} = -2(G^i - \frac{1}{4} F^i)$$

The solutions of this system can be seen as integrable curves of the vector field S^* on TM given by:

(1.4.)
$$S^* = y^i \frac{\partial}{\partial x^i} - 2G^{*i}(x,y)\frac{\partial}{\partial y^i}$$

$$G^{*i} = G^i - \frac{1}{4} F^i$$

and also is a semispray.

The semispray associated to Lagrangian L is

(1.4’)
$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$$

S is a vector field on the $\tilde{TM}$ which depends only of the MLS.

2. Dissipative control of MLS.

Definition 2.1. *The mechanical system* $\sum$ *is* ***dissipative*** *if the force F is dissipative i.e.* $F_j(x,y)y^j \le 0$ *and* $\sum$ *is* ***strict dissipative*** *if the force F is strict dissipative i.e.* $F_j(x,y)y^j \le -\alpha y_j y^j$ *, with* $\alpha > 0$ *and* $y_j = g_{ij}(x,y)y^i$.

The conditions of dissipativity and of strictly dissipativity can be also formulated as it follows:

If the matrix $(g_{ij}(x,y))$ is positively defined, it defines a Riemannian metric g in the vertical bundle over TM, by formulae

(2.1.) $$g(A,B) = g_{ij} A^i B^j$$

for vertical fields $A = A^i \frac{\partial}{\partial y^i}$ and $B = B^j \frac{\partial}{\partial y^j}$.

The force field (F^i) can be seen as a section in vertical bundle by definition $F = F^i(x,y)\frac{\partial}{\partial y^i}$.

The Liouville field $C = y^i \frac{\partial}{\partial y^i}$ appears as a section in vertical bundle and we have

(2.2.) $$g(C,C) = g_{ij} y^i y^j = y_i y^j := \| y \|^2$$

Also, we have $F_j(x,y)y^j = g(F,C)$ and so, the dissipativity condition one writes $g(F,C) \le 0$ and the strictly dissipation condition becomes $g(F,C) \le -\alpha \| y \|^2$,when the system is dissipative, the energy of the system decreases on the integral curves of the equations (1.3.):

Theorem 2.1. *If the Lagrangian system* $\sum$ *is dissipative, then its energy* $E(x,y) = y^i \frac{\partial L}{\partial y^i} - L$ *decreases on the solutions curves of the equations (1.3).*

If the system $\sum$ *is strictly dissipative and the solutions curves have not singularities, then the energy E is strictly decreasing on solutions curves of the equations (1.3).*

Proof. Let $\gamma : t \to (x(t), y(t)), y = \dot{x}$ a curve on TM, solution of the system $\nabla_{\dot{\gamma}} \dot{\gamma} = F \circ \gamma$, where we noted with ∇ Levi-Civita connexion of (M,g) manifold. Along this curve, we have:

$$\frac{dE}{dt} = \ddot{x}\frac{\partial L}{\partial \dot{x}^i} + \dot{x}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \ddot{x}\frac{\partial L}{\partial \dot{x}^i} - \dot{x}^i \frac{\partial L}{\partial x^i} - \ddot{x}\frac{\partial L}{\partial \dot{x}^i} =$$

$$= \dot{x}^i \left(\frac{d}{dt}\left(\frac{\partial L}{\partial x^i}\right) - \frac{\partial L}{\partial x^i} \right) = \dot{x}^i F_i(x,\dot{x}) \le 0$$

So E is decreasing on γ.

If $\sum$ is strictly disipative,then

$$\frac{dE}{dt} = \ddot{x}F_i(x,\dot{x}) \le -\alpha \| \dot{x} |^2 < 0$$

so E is strictly decreasing on γ.

We define the equilibrum point of the system $\sum$ as zeros of the semispray S*.

From (1.4.) it results that the equilibrum point of the system $\sum$ are $(x_0^i,0)$ or $O_{x_0} \in T_{x_0}M$, where (x_0^i) must be a solution of the equations

$$G^{*i}(x_0^i,0) = 0 \Leftrightarrow G^i(x_0^i,0) - \frac{1}{4}F^i(x_0^i,0) = 0$$

For a Lagrange manifold (M,L), the tangent manifold TM is a Riemannian manifold assuming that $g_{ij}(x,y) = \frac{\partial^2 L}{\partial y^i \partial y^j}$ is positively defined.

The Riemannian metric on TM is

$$g_L = g_{ij}dx^i \otimes dx^j + g_{ij}\delta y^i \delta y^j$$

with

$$\delta y^i = dy^i + N_j^i dx^i$$

where (N_j^i) are the coefficients of the nonlinear connection defined by the semispray $N_j^i = \frac{\partial G^i}{\partial y^j}$.

If the manifold (TM,G) is complete as a metric space one can be given theorems of stability for the equilibrum points of a vector field on TM, similar to those from R^n.

To give sufficient condition so that the system will be stable, we use a Lyapunov function [2] constructed with the energy of the system.

So, we have:

Let $\sum = (M, L(x,y), F(x,y))$ be a dissipative Lagrangian systems with (TM, g_L) a complete Riemannian manifold. Let $(x_0^i,0)$ be an equilibrum point of Σ, that is a zero of the vector field S*. We suppose that $(x_0^i,0)$ is a point of absolute minimum for the energy E of the system Σ. Then $(x_0^i,0)$ is a stable equilibrum point.

And also:

Let $\sum = (M, L(x,y), F(x,y))$ be a dissipative Lagrangian system with $(TM \;, g_L)$ a complete Riemannian manifold, L≥0 and L a homogenous function of degree $m \geq 2$ in the variables y. Let $(x_0^i, 0)$ be a point on TM with $F^i(x_0^i, 0) = 0$. Then, $(x_0^i, 0)$ is a stable point of S*.

Conclusion.

The energy of the system decreases on the integral curves.

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