Some Results on Fuzzy Theory

Angel Garrido
Faculty of Sciences, National University of Distance Education, Madrid, Spain
Paseo Senda del Rey, 9, 28040, Madrid, Spain
algbmv@telefonica.net

Abstract

The apparition of Fuzzy Logic [10] has had a double repercussion on scientific research, and has provoked two types of reactions. From a theoretical point of view, it is indeed a very useful generalization of the classical Set Theory proposed by Boole and Cantor, in this way making possible our analysis of uncertainty. But unfortunately, in his first steps it had to avoid the assaults of routine minds from the often too rigid mathematical field. This situation improved later, especially in nations with less deep-rooted prejudices. And by contrast, the new theory has obtained a strong rooting in nations with new and increasing scientific potential, such as China, Japan, and South Korea [9]. More recently it has also become rooted in European countries, such Hungary, Spain [6], and Romania, mainly due to its successful technical applications. We analyze some essential aspects of this new and powerful tool of Mathematical Analysis. This paper is based on our previous work [2], [3].

Keywords: Fuzzy Theory, Artificial Intelligence.

1. Introduction

When we solve problems in Artificial Intelligence, its representation will be through the Fuzzy Logic techniques, a very useful procedure. For instance, problems related to the "real world". As you know, it is only one of the "possible worlds". We define the *"world"* as *"a complete and consistent description of how things are or how they could have been"*. [1],[4],[5],[7],[8].

In such type of questions Monotonic Logic often does not work, whereas such a type of Logic is the classical one in formal worlds, such as in Mathematics.

Also, another Non-Monotonic Logics must be introduced, where now the extension of the set of sentences can modify the conclusion. This happens frequently in the real world: for instance, in medical sciences, or in the common sense reasoning, with partial information, giving temporal, revisable and provisional conclusions.

We need Fuzzy Sets, Fuzzy Relations and so on, to describe the gradation of certainty in our world. The aforementioned membership function, μ, is shown through a new function, which describes the degree of fulfillment for each element of the property defining the set, or equivalently the degree of mutual relation between every couple of elements. Such "membership degree" value can be assigned by the corresponding μ, the "membership function", whose range is the closed unit interval [0, 1].

So, the application can be denoted

$$\mu : C \rightarrow [0,1]$$

According to this, a fuzzy set can be defined by

$$C = \{x \mid \mu(x), \forall x \in U\}$$

Where the vertical symbol "|" does not mean "such that", but it adjoins the information on the "membership degree" of such element to the set C.

Let $\{U_i\}_{i=1\ldots n}$ be *n* universes of discourse. We define a fuzzy relation, R, through a membership function that associates each n-uple, $\{x_i\}_{i=1\ldots n}$, where $x_i \in U_i$, i = 1, 2, ..., n, with a value in the unit closed interval, [0, 1],

$$\{x_i\}_{i=1\ldots n} \in \prod U_i \rightarrow r \in [0,1]$$

$$(x_1, x_2, ..., x_n) \rightarrow \mu\,(x_1, x_2, ..., x_n) = r : 0 \leq r \leq 1$$

The fuzzy relation, *R,* can be defined through such "membership function", μ. In this manner, we will have gradation in the relationship,

$$0\ R, ..., (1/3)\ R, ..., (1/2)\ R, ..., (2/3)\ R, ..., 1\ R$$

The Cartesian product of two fuzzy sets, F (in the universe U_1) and G (in the universe U_2), is the subsequent fuzzy binary relation

$$F \times G = \{(x, y) \mid \mu_{F \times G}(x, y) = \min [\mu_F(x), \mu_G(y)], \forall x \in U_1, \forall y \in U_2\}$$

As you know, we can consider the precedent fuzzy relation as a subset of the adequate Cartesian product,

$$R \subset F \times G$$

The composition of fuzzy relations can be defined by

$$R_1 (U_1, U_2) \circ R_2 (U_2, U_3) = R_3 (U_1, U_3)$$

where

$$R_3 (U_1, U_3) = \{(x, z) \mid \mu_{R1 \circ R2}(x, z), \forall x \in U_1, \forall z \in U_3\}$$

So,

$$\mu_{R1 \circ R2}(x, z) = \max \{\forall y \in U_2: \min (\mu_{R1}(x, y), \mu_{R2}(y, z))\} = \max \{\min (\mu_{R1}(x, y), \mu_{R2}(y, z))\}$$

There exists a clear analogy between the composition of fuzzy relations and the matrix product. For this reason, the composition ($\circ$) of fuzzy relations can also be denominated as the *"max-min matrix product"*.

As a particular case of the previous operation of composition between fuzzy relations, we can introduce the composition between a fuzzy set and a fuzzy relation. Obviously, in such a case, the fuzzy set can be represented by a row or column matrix. These can be very useful in "Fuzzy Inference".

2. The Non-Boolean Algebra of Fuzzy Sets

We can introduce new generalized versions of the Classical Logic. This can be done through the Generalized Modus Ponens or the Generalized Modus Tollens, and also by the Hypothetic Syllogism.

To each Fuzzy Predicate, we can associate a Fuzzy Set, defined by such a property, that is, composed by the elements of the universe of discourse such that they totally or partially verify such a condition. So, we can prove this:

The class of Fuzzy Sets, with the operations, $\cup$, $\cap$ *and c* (being c the pass to the complement) *does not constitute a Boolean Algebra*

It is because neither the Contradiction Law nor the Third Excluded Principle works in it. Both proofs can be expressed easily, in algebraic or geometrical way.

3. Difference of Fuzzy Sets

If we take two sets, A and B, the difference is given by

$$A - B = A \cap c\,(B)$$

There exist two means of obtaining the difference between fuzzy sets:

- By simple method: For instance, if we take:

$$A = \{a|0.1, b|0.3, c|0.6, d|0.9\}$$

and

$$B = \{a|0.2, b|0.5, c|0.8, d|1\}$$

then

$$c\,(B) = \{a|0.8, b|0.5, c|0.2, d|0\}$$

Therefore

$$A - B = A \cap c\,(B) = \{a|0.1, b|0.3, c|0.2, d|0\}$$

While the *Bounded Difference* is defined through a new operator, θ, according to the membership function,

$$\mu_{A\,\theta\,B}(x) = \max \{\mu_A(x) - \mu_B(x), 0\}$$

It is clear that it does not verify the commutative property, because in the previous example,

$$B\,\theta\,A = \{a|0.1, b|0.2, c|0.2, d|0.1\} \neq A\,\theta\,B$$

To introduce the distance between fuzzy sets, A and B, we can consider different possibilities, now based on the values of the membership functions on the point $x \in U$,

1) the well-known *Euclidean distance,*

$$e(A, B) = [\sum \{\mu_A(x) - \mu_B(x)\}^2]^{1/2}$$

2) the *Hamming distance,*

$$d(A, B) = \sum |\mu_A(x_i) - \mu_B(x_i)|$$

with $i \in \{1, 2, ..., n\}$, and $x_i \in U$.

We can easily prove the four conditions to be distance. And also, the relative Hamming distance (δ)can be defined, when the universal set U is finite, for instance, with *n* elements, if card (U) = n, then

$$\delta(A, B) = (1/n)\, d(A, B)$$

For instance, let A and B be as in the aforementioned example. Thus,

$$e(A, B) = 0.316$$

$$d(A, B) = 0.6$$

$$\delta(A, B) = (1/n)\, d(A, B)$$

So, if n = 4, then

$$\delta(A, B) = (1/4)\, d(A, B) = 0.15$$

And by generalizing, we can also define the Minkowski distance,

$$d_w(A, B) = [\sum |\mu_A(x) - \mu_B(x)|^{w}\}]^{1/w}, \text{ with } w \in [1,+\infty]$$

Note that when m = 1, we obtain the Hamming distance. And when m = 2, we find the Euclidean distance. Both are particular cases, therefore, of such more general Minkowskian distance.

4. Fuzzy Distance Between Fuzzy Sets

We need to introduce the Extension Principle, according to which if we depart from a Cartesian product of universal sets:

$$\mathbf{U} = \prod U_i,\ i = 1, 2,..., r$$

And a collection of fuzzy sets, each one into the corresponding universal set,

$$A_i \in U_i,\ i = 1, 2,..., r$$

Then, we define the Cartesian product of fuzzy sets,

$$\prod A_i,\ i = 1, 2, ..., r$$

by its membership function,

$$\mu_{\prod Ai}(x_1, x_2, ..., x_r) \equiv \min \{\mu_{A1}(x_1), \mu_{A2}(x_2), ..., \mu_{Ar}(x_r)\}$$

Let F be the function from the universe U to the universe V. Then, the fuzzy set $B \subseteq V$ can be obtained by F and the collection of fuzzy sets, $\{A_i\}_{i=1,2,\dots, r}$, in this way:

$$\mu_B(y) = 0, \text{ if } F^{-1}(y) = \emptyset$$

and

$$\mu_B(y) = \max [\min \{\mu_{A1}(x_1), \mu_{A2}(x_2), ..., \mu_{Ar}(x_r)], \text{ if } F^{-1}(y) \neq \emptyset$$

If the function F is one-to-one, we have

$$\mu_B(y) = \mu_A(F^{-1}[y]), \text{ if } F^{-1}(y) \neq \emptyset$$

Let (U, d) be a pseudo-metric space. Therefore, with

$$d: UxU \to R_+ \cup \{0\}$$

so that it verifies

1) $d(x, x) = 0, \forall x \in U$
2) $d(x, y) = d(y, x), \forall x, y \in U$
3) $d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in U$

Remember also that with the additional condition

4) if $d(x, y) = 0$, then $x = y$

it turns *d* into a distance, and in such a case, *(U, d)* will be a *metric space.*

In our pseudo-metric space, (U, d), if we take two fuzzy subsets, A and B, it is possible to introduce by the extension principle the pseudo-metric distance between A and B:

$$\forall \rho \in \mathbf{R}_{+}, \mu_{d(A, B)}(\rho) = \max [\min \{\mu_A(a), \mu_B(b)\}]$$

And it will also be a fuzzy set.

5. Conclusion

With these considerations about some special functions in AI we have revealed a more possible approximation to the problems of AI.

But also the introduction of fuzzy measures offers us a parallel and very fructiferous alternative way to the Classical Measure Theory, generalizing and so, improving many times very well known results of the Mathematical Analysis, as may be the Lusin Theorem, the Egorov Theorem, the Hahn-Banach Theorem, and many practical results that belongs to Combinatorial and Fuzzy Optimization, etc. This provide a more realistic approach to many problems related with new scientific aspects into the mathematical research, between them may be mentioned the treatment of uncertainty, in data mining, natural-language processing, and so one.

All them, are greatly interrelated with the modern Graph Theory and the Probability Calculus, which for instance provides us very useful tools, as the Probabilistic Graphical Models, modulating by their nodes and edges, through the associated distributions, which also belongs to the Mathematical Analysis.

And also must be appreciated the very practical and increasing theoretical developments provided by Fuzzy Theory, in many different problems, as may be the Rule-Based Systems (RBs), which in particular admits to be modeling fuzzifying the Basis of Facts, and / or the Basis of Knowledge.

Obviously, it will be necessary to analyze in detail what results continue being valid (perhaps with little, but very subtle modifications), and which not of those which come from the Classic Analysis.

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