Black-Scholes formula - a Heston approach

Bogdan Patrut
Vasile Alecsandri University of Bacau, Faculty of Sciences, Romania
bogdan@edusoft.ro

Tiberiu Socaciu
Stefan cel Mare University of Suceava, Faculty of Economics and Public Administration, Romania
tibisocaciu@yahoo.com

Abstract
In this paper we will compare Black-Scholes formula with a particular case of Heston formula, both solutions of the same problem.

Keywords: Black-Scholes model, Heston model, comparing analytic solutions.

1. Introduction
As an extension of Black and Scholes (1973) model:

\[ dS = \mu S \, dt + \sigma^{0.5} S \, dW, \quad (1) \]

where
a) W is an Wiener process;
b) \( \mu \) is a constant named drift;
c) \( \sigma \) is a constant named volatility;
d) S is a process for a traded asset.

Steven and Heston (1993) define a new model with a stochastic volatility, see equation (2):

\[ dS = \mu S \, dt + \sqrt{\nu} S \, dW \\
\frac{d
u}{dt} = \theta (\sigma - \nu) \, dt + \xi \nu^{0.5} \, dB, \quad (2) \]

where:
a) \( \omega \) is long term of volatility; 
b) \( \theta \) is return factor to mean of volatility (\( \sigma \)); 
c) \( \xi \) is volatility of volatility; 
d) B and W are Wiener standard processes \( \rho \)-correlated; 
e) S is a stochastic process for a traded asset; 
f) \( \nu \) is a stochastic process for volatility.

This model was extended by Christoffersen, Heston and Jacobs (2009) as a model with two stochastic semi-volatilities. In our opinion, this model can be generalized as a stochastic model with \( q \) (\( q>0 \)) stochastic partial-(or semi-)volatilities.

2. Solutions of tho models in mirror

<table>
<thead>
<tr>
<th>SDE for BS model</th>
<th>( dS = \mu S , dt + \sigma^{0.5} S , dW )</th>
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</thead>
<tbody>
<tr>
<td>Analytic solutions for european calls</td>
<td>( V(s,t) = N(d_1) S - N(d_2) E \exp(-r (T-t)) )</td>
</tr>
<tr>
<td>( d_1 = \sigma^{1} (T-t)^{0.5} [\ln(S/E) + (r + 0.5 \sigma^2)(T-t)] )</td>
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<tr>
<td>( d_2 = \sigma^{1} (T-t)^{0.5} [\ln(S/E) + (r - 0.5 \sigma^2)(T-t)] )</td>
<td></td>
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</tbody>
</table>
with E strike with BS model

| $N(x) – pdf of N(0,1)$ distribution |

Official references for model and analytic solutions for BS model

| (Black, F. & Scholes, M., 1973) |

SDEs for H model

| $dS = \mu S dt + \sqrt{v} S dW$
| $dv = \theta (\sigma - v) dt + \xi \sqrt{v} dB$ |

Analytic solutions for European calls with E strike with H model

| $V(s, v, t) = S P_1 - E \exp(-r(T-t)) P_2$
| $P_j = \frac{1}{2} + \frac{\varphi}{\pi} \int \frac{\exp(i \varphi \log(E)) f_j(x, v, t, \varphi)}{i \varphi} d\varphi$
| $f_j(x, v, t, z) = \exp(C_j(T-t, z) + D_j(T-t, z) z + i z x)$
| $C_j(t, z) = rz_i t + a [(b_1 - \rho \xi z i + d_j) t - 2 \log(1 - g_j \exp(d_j r))]$
| $D_j(t, z) = [(b_2 - \rho \xi z i + d_j) [1 - \exp(d_j r))] \xi^2 [1 - \exp(d_j r))]^{-1}$
| $g_j = [(b_1 - \rho \xi z i + d_j) [1 - \exp(d_j r))] \xi^2 [1 - \exp(d_j r))]^{-1}$
| $d_j = [(b_2 - \rho \xi z i + d_j) [1 - \exp(d_j r))] \xi^2 [1 - \exp(d_j r))]^{-1}$
| $u_1 = \frac{1}{2}$
| $u_2 = -\frac{1}{2}$
| $a = k \theta$
| $b_1 = k + \lambda - \rho \xi$
| $b_2 = k + \lambda$
| $j = 1, 2$ |

Official references for model and analytic solutions for H model

| (Steven & Heston, 1993) |

3. Links between solutions?

We can point that Heston model is a generalization of Black-Scholes model. For

\[
\begin{align*}
v &= \sigma \quad (3) \\
dv &= 0 \quad (4)
\end{align*}
\]

the two models are identical.

But

\[
\begin{align*}
dv &= 0 \quad (5)
\end{align*}
\]

is same with

\[
\theta = \xi = 0, \quad (6)
\]

that means:
\[ a = k \theta = 0, \quad (7) \]

\[ b_1 = k + \lambda - \rho \xi = k + \lambda = b_2, \quad (8) \]

\[ d_j = (b_j - \rho \xi z_i)^2 - \xi^2 (2 u_j z_i - z_i^2))^{0.5} = (b_j - \rho \xi z_i)^2 \xi^{0.5} = |b_j|, \quad (9) \]

\[ g_j = [b_j - \rho \xi z_i + d_j] [b_j - \rho \xi z_i - d_j]^{-1} = [b_j + |b_j|] [b_j - |b_j|]^{-1} = 0, \quad (10) \]

if assume that

\[ k + \lambda < 0 \]

\[ D_j(t, z) = [(b_j - \rho \xi z_i + d_j) [1 - \exp(d_j r))] \xi^2 [1 - \exp(\xi)]^{-1} = [b_j + |b_j|] [1 - \exp(|b_j| r))] \xi^2 [1 - \exp(|b_j| r))]^{-1} = 0, \quad (11) \]

if assume that

\[ [b_j + |b_j|] \xi^2 = 0/0 = 0, \quad (12) \]

\[ C_j(t, z) = r z i t + a [(b_j - \rho \xi z_i + d_j) t - 2 \log(1 - g_j \exp(d_j r)) + 2 \log(1 - g_j)] \xi^2 = r z i t, \quad (13) \]

if assume that

\[ + a \xi^2 = 0/0 = 0. \quad (14) \]

\[ f_j(x, v, t, z) = \exp(C_j(T - t, z) + D_j(T - t, z) z + i z x) = \exp(r z i t + i z x) \]

\[ P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\exp(i x \log(E)) f_j(x, v, t, \varphi)}{i \varphi} \, d\varphi = \frac{1}{2} + \int_0^\infty \frac{\exp(i x \log(E)) \exp(r z i t + i x)}{i \varphi} \, d\varphi \]

\[ + i z x) i^{-1} z^{-1} \right] dz = \frac{1}{2} + \int_0^\infty \cos(z \log(E)) \sin(z \log(E)) \cos(r z t + z x) \]

\[ + i \sin(r z t + z x) i^{-1} z^{-1} \right] dz = \frac{1}{2} + \int_0^\infty \sin(z \log(E)) \cos(r z t + z x) + \sin(z \log(E)) \cos(r z t + z x) \]

\[ z^{-1} \right] dz = \frac{1}{2} + \int_0^\infty \sin(z \log(E) + r z t + z x) \right] z^{-1} \left. dz. \quad (15) \]

4. Comments and further works

We expected that the two solutions are identical. Because not getting the same result on the two different routes, results that Heston solution has a little inconsistency on some particular cases, like \( \xi = 0 \) (we use that 0/\( \xi = 0! \)). Therefore, a revision of the solution Heston by treating individual cases. We intend to do so in the future.

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